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Una Solución de la Pep N° 1
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(1) Si $w = f(x, y)$ es una función cuyas derivadas parciales existen y además,

$$\begin{cases} x = e^{\alpha}\cos\beta \\ y = e^{\alpha}\sin\beta \end{cases}$$

Entonces demuestre que

$$e^{2\alpha} = \frac{\left(\frac{\partial f}{\partial \alpha}\right)^2 + \left(\frac{\partial f}{\partial \beta}\right)^2}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

Una Solución: Inicialmente observamos que

$$e^{2\alpha} = \frac{\left(\frac{\partial f}{\partial \alpha}\right)^2 + \left(\frac{\partial f}{\partial \beta}\right)^2}{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \Longleftrightarrow e^{2\alpha}\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right) = \left(\frac{\partial f}{\partial \alpha}\right)^2 + \left(\frac{\partial f}{\partial \beta}\right)^2$$

A continuación aplicamos la regla de la cadena a la función f.

$$\begin{aligned}
\frac{\partial f}{\partial \alpha} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} = \frac{\partial f}{\partial x}e^{\alpha}\cos\beta + \frac{\partial f}{\partial y}e^{\alpha}\operatorname{sen}\beta \Longrightarrow \\
\left(\frac{\partial f}{\partial \alpha}\right)^2 &= \left(\frac{\partial f}{\partial x}e^{\alpha}\cos\beta + \frac{\partial f}{\partial y}e^{\alpha}\operatorname{sen}\beta\right)^2 \\
&= \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}\cos^2\beta + 2\frac{\partial f}{\partial x}e^{\alpha}\cos\beta\frac{\partial f}{\partial y}e^{\alpha}\operatorname{sen}\beta + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}\operatorname{sen}^2\beta \\
&= \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}\cos^2\beta + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}e^{2\alpha}\cos\beta\operatorname{sen}\beta + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}\operatorname{sen}^2\beta \qquad (*)
\end{aligned}$$

Análogamente

$$\begin{aligned}
\frac{\partial f}{\partial \beta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \beta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \beta} = \frac{\partial f}{\partial x}(-e^{\alpha}\operatorname{sen}\beta) + \frac{\partial f}{\partial y}e^{\alpha}\cos\beta \Longrightarrow \\
\left(\frac{\partial f}{\partial \beta}\right)^2 &= \left(\frac{\partial f}{\partial x}(-e^{\alpha}\operatorname{sen}\beta) + \frac{\partial f}{\partial y}e^{\alpha}\cos\beta\right)^2 \\
&= \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}\operatorname{sen}^2\beta + 2\frac{\partial f}{\partial x}(-e^{\alpha}\operatorname{sen}\beta)\frac{\partial f}{\partial y}e^{\alpha}\cos\beta + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}\cos^2\beta \\
&= \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}\operatorname{sen}^2\beta - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}e^{2\alpha}\operatorname{sen}\beta\cos\beta + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}\cos^2\beta \qquad (**)
\end{aligned}$$

Finalmente de $(*)$ y $(**)$ sigue que

$$\begin{aligned}
\left(\frac{\partial f}{\partial \alpha}\right)^2 + \left(\frac{\partial f}{\partial \beta}\right)^2 &= \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}\cos^2\beta + 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}e^{2\alpha}\cos\beta\operatorname{sen}\beta + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}\operatorname{sen}^2\beta + \\
&\quad \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}\operatorname{sen}^2\beta - 2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}e^{2\alpha}\operatorname{sen}\beta\cos\beta + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}\cos^2\beta \\
&= \left(\frac{\partial f}{\partial x}\right)^2 e^{2\alpha}(\cos^2\beta + \operatorname{sen}^2\beta) + \left(\frac{\partial f}{\partial y}\right)^2 e^{2\alpha}(\operatorname{sen}^2\beta + \cos^2\beta) \\
&= e^{2\alpha}\left(\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right)
\end{aligned}$$

[1]Cada problema vale 2.0 puntos
Tiempo 90'

(2) Determine y' si

$$\frac{x^3y^4 + x^2e^{x+y}}{6 - \cos(xy)} = 1$$

Una solución: Podemos proceder como sigue:

$$\begin{aligned} \frac{x^3y^4 + x^2e^{x-y}}{6 - \cos(xy)} = 1 \quad &\Longleftrightarrow \quad x^3y^4 + x^2e^{x+y} = 6 - \cos(xy) \\ &\Longleftrightarrow \quad x^3y^4 + x^2e^{x+y} + \cos(xy) - 6 = 0 \end{aligned}$$

Así que

$$y' \quad = \quad -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2y^4 + 2xe^{x+y} + x^2e^{x+y} - y\,\mathrm{sen}(xy)}{4x^3y^3 + x^2e^{x+y} - x\,\mathrm{sen}(xy)}$$

(3) Determine máximos, mínimos y puntos silla, si existen, para las función

$$f(x,y) = 36xy - x^3 - y^3 - 1$$

Solución: Determinamos las derivadas parciales de la función f.

$$\frac{\partial f}{\partial x} = 36y - 3x^2$$
$$\frac{\partial f}{\partial y} = 36x - 3y^2$$

Ahora determinamos los puntos críticos:

$$\left.\begin{array}{rcl} 36y - 3x^2 & = & 0 \\ 36x - 3y^2 & = & 0 \end{array}\right| \Longrightarrow \left.\begin{array}{rcl} 12y - x^2 & = & 0 \\ 12x - y^2 & = & 0 \end{array}\right| \Longrightarrow x^2 - y^2 + 12x - 12y = 0 \Longrightarrow (x-y)(x+y) + 12(x-y) = 0$$
$$\Longrightarrow \quad (x-y)[x+y+12] = 0 \Longrightarrow y = x \quad \vee \quad y = -12 - x$$

Luego, tenemos que si:

$$y = x \quad \Longrightarrow \quad 12x - x^2 = 0 \Longrightarrow x(12-x) = 0 \Longrightarrow x = 0 \vee x = 12$$

Por tanto tenemos los puntos críticos $P = (0,0)$ y $Q = (12,12)$ Además si

$$\begin{aligned} y = -12 - x \quad &\Longrightarrow \quad 12(-12-x) - x^2 = 0 \Longrightarrow -144 - 12x - x^2 = 0 \Longrightarrow x^2 + 12x + 144 = 0 \\ &\Longrightarrow \quad x = \frac{-12 \pm \sqrt{144 - 4(144)}}{2} = \frac{-12 \pm 12\sqrt{-3}}{2} \notin \mathbb{R} \end{aligned}$$

Para estudiar la calidad de P Q construimos $\Delta(x,y)$, y para ello determinamos las segundas derivadas parciales:

$$\frac{\partial^2 f}{\partial x^2} = -6x$$
$$\frac{\partial^2 f}{\partial y^2} = -6y$$
$$\frac{\partial^2 f}{\partial y \partial x} = 36$$

Luego,

$$\Delta(x,y) \quad = \quad 36xy - (36)^2$$

Así que para.
$P = (0,0)$ tenemos que $\Delta(0,0) = -(36)^2 < 0$ es un punto silla,y para

$Q = (12,12)$ tenemos que $\Delta(12,12) = 36 \cdot 144 - (36)^2 > 0$ y $\dfrac{\partial^2 f}{\partial x^2}(12,12) = -6 \cdot 12 < 0$, por tanto Q es un máximo.